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# OPTIMAL STABILIZATION OF MOTION WITH RESPECT TO SOME OF THE VARIABLES*

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The use of non-linear transformations of variables and the theory of implicit functions yield sufficient conditions for the stabilization of the unperturbed motion of a certain class of non-linear control systems. Upon stabilization one can achieve Lyapunov stability and asymptotic stability with respect to some of the variables. Closed formulae are obtained which make it possible to organize an iterative process which will determine the most satisfactory (optimal) stabilization law from the practical point of view. A technique is worked out by which the construction of control laws in the original non-linear system can be reduced to the construction of control laws for an auxiliary linear control system of a simpler type. This technique is very similar to a principle that has become quite popular in the modern applied theory of automatic control - the iterative construction of optimal control laws. As an application, the technique is used to stabilize the equilibrium position of a rigid body by means of Cardan-suspended gyroscopes and motors in which the tractive force is continuously regulated.

*1. Statement of the problem.* Suppose the perturbed motion of the object being controlled is described by a non-linear system of ordinary differential equations

$$\mathbf{x}^{\cdot} = \mathbf{X}(t, \mathbf{x}, \mathbf{u}), \quad \mathbf{X}(t, 0, 0) \equiv 0, \quad \mathbf{x} \in R^n, \quad \mathbf{u} \in R^r \tag{1.1}$$

whose right-hand sides are defined and continuous together with their partial derivatives with respect to $t, \mathbf{x}, \mathbf{u}$ of up to second order inclusive in the domain

$$t \geqslant 0, \quad \|\mathbf{x}\| \leqslant H, \quad \|\mathbf{u}\| < +\infty \tag{1.2}$$

The control vector will be sought in the class of vector-valued functions $\mathbf{u} = \mathbf{u}(t, \mathbf{x})$, $\mathbf{u}(t, 0) \equiv 0$ which are continuously differentiable in the domain $D$: $t \geqslant 0, \|\mathbf{x}\| \leqslant H$. In that case the right-hand sides of the closed system (1.1) satisfy the conditions of the existence and uniqueness theorem for $D$.

Our task is to choose the vector $\mathbf{u} = \mathbf{u}(t, \mathbf{x})$ in such a way /1-6/ that 1) the unperturbed motion $\mathbf{x} = 0$ of system (1.1) is stable with respect to all variables in Lyapunov's sense and asymptotically stable with respect to a certain specified subset of the variables characterizing it; and 2) a certain functional, characterizing the transient in the system and the output of the controls expended in the stabilization process, is minimized along the trajectories of system (1.1).

If one is guided in the choice of the objective functional by the initial technical requirements only, the chances of finding a solution of the stabilization problem in a rigorous, closed form become severely limited. However, in the rare cases when such solutions are found, they still need correction, because many technical requirements are mutally contradictory and not always amenable to formalization. In applications, therefore, the most practical approach to the solution of optimal stabilization problems will involve an iterative procedure /3, 4/. In other words: one first (at the planning stage) makes a preliminary choice of the objective functional and then (still at the planning stage, or perhaps already at the operational stage) corrects it iteratively - by readjusting coefficients or sometimes even modifying the structure of the functional itself. The real-time implementation of such procedures requires that each individual step of the solution procedure be as simple as possible.

The technique proposed below for solving optimal stabilization problems agrees on the whole with the approaches used in /1-6/, the only difference being that it reduces the construction of optimal control laws for the original non-linear system to an analogous construction for a simpler, auxiliary linear control system, derived from the original system by a non-linear transformation of variables /7, 8/.

*2. The class of systems (1.1) under consideration.* We shall assume that the controls applied to the system are confined to one specific group of equations in system (1.1) and that Eqs.(1.1) have the form

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$$\mathbf{w}^{\cdot} = \mathbf{X}^{(0)}(\mathbf{w}, \xi, \eta), \quad \xi^{\cdot} = \mathbf{X}^{(1)}(\mathbf{w}, \xi, \eta, \mathbf{u}), \quad \eta^{\cdot} = \mathbf{X}^{(2)}(\xi, \eta) \tag{2.1}$$
$$\mathbf{w}, \mathbf{X}^{(0)} \in R^m; \quad \xi, \mathbf{X}^{(1)} \in R^l; \quad \eta \mathbf{X}^{(2)} \in R^{n-m-l}; \quad \mathbf{x} = (\mathbf{w}, \xi, \eta)$$

We shall assume that $n - m - l \geqslant r$ and, in addition, that the variables $\xi, \eta$ are related as follows:

$$\mathbf{Y}(\xi, \eta) = 0, \quad \mathbf{Y}(0, 0) = 0, \quad \mathbf{Y} \in R^k, \quad k \geqslant n - m - 2r \tag{2.2}$$

where the components of the vector-valued function $\mathbf{Y}$ are continuously differentiable with respect to $\xi, \eta$ in the domain (1.2). In the case $2r \geqslant n - m$ one can consider a system (2.1) in which the phase variables $\xi, \eta$ are not necessarily related by the additional constraints (2.2).

3. ***Auxiliary Jacobians.*** Consider the functions

$$\Phi_i(\mathbf{w}, \xi, \eta, \mathbf{u}) = \sum_{j=1}^{l} \frac{\partial X_i^{(2)}}{\partial \xi_j} X_j^{(1)} + \sum_{s=1}^{n-m-l} \frac{\partial X_i^{(2)}}{\partial \eta_s} X_s^{(2)} \tag{3.1}$$
$$(i = 1, \ldots, n - m - l)$$

and introduce the Jacobian $F(\mathbf{x}, \mathbf{u}) = (\partial \Phi_i / \partial u_j)$ of the functions $\Phi_i$ $(i = 1, \ldots, n - m - l)$ with respect to the variables $u_j$ $(j = 1, \ldots, r)$. Let $\operatorname{rank} F(\mathbf{0}, \mathbf{0}) = r$ and assume that the first $r$ rows of $F(0, 0)$ are linearly independent (this assumption involves no loss of generality). Put $\eta = (\rho, \mu)$, $\rho \in R^r$, $\eta \in R^{n-m-l-r}$ and introduce the Jacobian $\Psi(\xi, \eta)$ of the functions $X_i^{(2)}$ $(i = 1, \ldots, r)$, $Y_j$ $(j = 1, \ldots, k)$ with respect to $\xi, \eta$.

4. ***Auxiliary linear system.*** Consider the equalities

$$\Phi_i(\mathbf{x}, \mathbf{u}) = u_i^{\wedge} \ (i = 1, \ldots, n - m - l) \tag{4.1}$$

in which the functions $\Phi_i$ have the form (3.1). Since $\operatorname{rank} F(0, 0) = r$, it follows from the theory of implicit functions that in the neighbourhood of the point $\mathbf{x} = 0$, $\mathbf{u} = 0$ for all $u_i^{\wedge}$ (here and below, unless otherwise specified, $i = 1, \ldots, r$), there exists a solution

$$\mathbf{u} = \mathbf{f}(\mathbf{x}, \mathbf{u}^{\wedge}), \quad \mathbf{f} \in R^r, \mathbf{u}^{\wedge} = (u_1^{\wedge}, \ldots, u_r^{\wedge}) \tag{4.2}$$

of system (4.1) in which the vector-valued function $\mathbf{f}$ is continuously differentiable in the neighbourhood of $\mathbf{x} = 0$ for all $u_i^{\wedge}$, and moreover $\mathbf{f}(0, 0) = 0$.

A direct calculation shows that $\eta^{\cdot\cdot} = \Phi(\mathbf{x}, \mathbf{u})$, and since by assumption $\operatorname{rank} F(0, 0) = r$, there exists a neighbourhood of the point $\mathbf{x} = 0, \mathbf{u} = 0$ in which, from the closed system (2.1), (4.2), we can isolate a system of linear equations

$$\rho_i^{\cdot\cdot} = u_i^{\wedge} \tag{4.3}$$

5. ***An auxiliary optimal stabilization problem.*** Using dynamic programming methods /9/, one can show that if

$$u_i^{\wedge} = c_i^{-1}(d_i \rho_i + d_{i+1} \rho_i^{\cdot}), \quad d_i = -\sqrt{a_i c_i} \tag{5.1}$$
$$d_{i+1} = -\sqrt{(b_i - 2d_i) c_i}$$

then the equilibrium position $\rho_i = \rho_i^{\cdot} = 0$ of each of subsystems (4.3) is asymptotically stable in Lyapunov's sense and a functional of the form

$$I_i = \int_{t_0}^{\infty} [a_i \rho_i^2 + b_i \dot{\rho}_i^2 + c_i u_i^{\wedge 2}]\, dt \tag{5.2}$$
$$a_i > 0, \quad b_i > 0, \quad c_i > 0$$

is minimized on the trajectories of each subsystem.

The constants $a_i, b_i, c_i$ in (5.2) are not fixed, since, as will be shown below, their optimum values are determined by an iterative procedure designed to find a solution of the original non-linear optimal stabilization problem which is acceptable from the practical point of view.

6. ***The main result.*** Considering the right-hand side of the first group of equations in the closed system (2.1), (4.2), (5.1), let us isolate the terms that depend only on $\mathbf{w}$, i.e., express the vector-valued function $\mathbf{X}^{(0)}$ in the domain $\|\mathbf{x}\| \leqslant H$ in the form

$$\mathbf{X}^{(0)}(\mathbf{w}, \xi, \eta) = \overline{\mathbf{X}}^{(0)}(\mathbf{w}) + \mathbf{X}_*^{(0)}(\mathbf{w}, \xi, \eta), \quad \mathbf{X}_*^{(0)}(\mathbf{w}, 0, 0) \equiv 0 \tag{6.1}$$

*Theorem 1* (stabilization of the unperturbed motion of system (2.1)). Assume that the following conditions hold:

1. rank $F(0, 0) = r$; 2°. rank $\Psi(0, 0) = n - m - r$

Then, if the trivial solution $\mathbf{w} = 0$ of the "truncated" system

$$\mathbf{w}^{\cdot} = \overline{\mathbf{X}}^{(0)}(\mathbf{w}) \tag{6.2}$$

is stable in Lyapunov's sense, the unperturbed solution $\mathbf{x} = 0$ of the closed system (2.1), (4.2), (5.1) is stable in Lyapunov's sense and asymptotically stable with respect to $\xi, \eta$. When that is the case the following functional is minimized along the trajectories of system (2.1), (4.2), (5.1):

$$I = \int_{t_0}^{\infty} \left\{ \sum_{i=1}^{r} [a_i x_{m+l+i}^2 + b_i (X_i^{(2)}(\xi, \eta))^2 + c_i \Phi_i^2(\mathbf{x}, \mathbf{u})] \right\} dt \tag{6.3}$$

*Proof*. By condition 2 and the theory of implicit functions, the equalities $X_i^{(2)} = 0$, $Y_j = 0$ $(j = 1, \ldots, k)$ have solutions

$$x_i = \varphi_i(\rho, \rho^{\cdot}), \quad \varphi_i(0, 0) = 0 \tag{6.4}$$
$$(i = m+1, \ldots, n;\ i \neq m+l+1, \ldots, m+l+r)$$

in a neighbourhood of the point $\xi = 0$, $\eta = 0$, where $\varphi_i$ are continuously differentiable with respect to $\rho, \rho^{\cdot}$. Eqs.(6.4) exhibit the relations between the variables occurring in the vectors $\xi, \eta$ and the variables $\rho, \rho^{\cdot}$.

Under a non-linear transformation of variables which enables one to split off a linear system (4.3) from the original system (2.1), the closed system (2.1), (4.2), (5.1) becomes

$$\mathbf{w}^{\cdot} = X^{(0)}(\mathbf{w}, \xi, \eta), \quad \xi^{\cdot} = \mathbf{X}^{(1)}(\mathbf{w}, \xi, \eta, \mathbf{f}(\mathbf{w}, \xi, \eta, \mathbf{u}^{\wedge})) \tag{6.5}$$
$$\eta^{\cdot} = \mathbf{X}^{(2)}(\xi, \eta), \quad \rho_i^{\cdot\cdot} = u_i^{\wedge}, \quad u_i^{\wedge} = c_i^{-1}(d_i \rho_i + d_{i+1} \rho_i^{\cdot})$$

In view of (6.1) and (6.4), we conclude, expressing the variables $\xi, \eta$ in terms of $\rho, \rho^{\cdot}$, that in some neighbourhood of the point $\mathbf{x} = 0$ system (6.5) splits off the following system of equations:

$$\mathbf{w}^{\cdot} = \overline{\mathbf{X}}^{(0)}(\mathbf{w}) + \overline{\mathbf{X}}_*^{(0)}(\mathbf{w}, \rho, \rho^{\cdot}), \quad \rho_i^{\cdot\cdot} = c_i^{-1}(d_i \rho_i + d_{i+1} \rho_i^{\cdot}) \tag{6.6}$$

Since Eqs.(6.4) impose the conditions $\varphi_i(0, 0) = 0$ on the functions $\varphi_i$, it follows that the variables occurring in the vectors $\xi, \eta$ vanish if $\rho = \rho^{\cdot} = 0$. Thus the vector-valued function $\overline{\mathbf{X}}_*^{(0)}$ in system (6.6) satisfies the condition $\overline{\mathbf{X}}_*^{(0)}(\mathbf{w}, 0, 0) \equiv 0$. In addition, since the functions $\varphi_i$ are continuously differentiable, the right-hand sides of system (6.6) satisfy the assumptions of the existence and uniqueness theorem for solutions.

Noting that the trivial solution $\rho_i = \rho_i^{\cdot} = 0$ of system (4.3), (5.1) is asymptotically stable in Lyapunov's sense (exponentially), while the trivial solution $\mathbf{w} = 0$ of the "truncated" system (6.2) is stable in Lyapunov's sense, we conclude from the reduction principle /10/ that trivial solution $\mathbf{w} = 0, \rho = \rho^{\cdot} = 0$ of system (6.6) is stable in Lyapunov's sense and asymptotically stable with respect to the variables $\rho, \rho^{\cdot}$.

The variables occuring in $\mathbf{w}$ are unaffected by the transformation of system (2.1), (4.2), (5.1) that isolates system (6.6). Therefore, the unperturbed solution $\mathbf{x} = 0$ of system (2.1), (4.2), (5.1) is stable with respect to $\mathbf{w}$. The variables $x_{m+l+i}$ of system (2.1), (4.2), (5.1) that occur in $\rho$ are also unaffected by this transformation of system (2.1), (4.2), (5.1). It follows from these conclusions and from (6.4) that the unperturbed solution $\mathbf{x} = 0$ of system (2.1), (4.2), (5.1) is asymptotically stable with respect to the variables $\xi, \eta$.

Thus, the unperturbed motion $\mathbf{x} = 0$ of the closed system (2.1), (4.2), (5.1) is stable in Lyapunov's sense and asymptotically stable with respect to the variables $\xi, \eta$.

Since system (4.3), (5.1) splits into $r$ independent subsystems, we can state not only that the functionals $I_i$ are minimized on its trajectories, but also that the same holds for their sum $I$. By equalities (4.1), $X_i^{(2)} = \rho_i^{\cdot}$, and the additivity of the integral, the functional $I$ has the form of (6.3). System (4.3), (5.1) is obtained from (2.1), (4.2) and (5.1) by a non-linear transformation of variables and consequently the functional (6.3) is minimized along the trajectories of system (2.1), (4.2) and (5.1). This completes the proof of the theorem.

Let us determine the domain of attraction of the unperturbed motion $\mathbf{x} = 0$ of system (2.1), (4.2), (5.1). Assume that condition 1 holds in the domain

$$\|\mathbf{x}\| \leqslant H_1, \quad \|\mathbf{u}\| \leqslant H_2 \tag{6.7}$$

and condition 2 in the domain $\|\xi^*\| \leqslant H_3$, $\xi^* = (\xi, \eta)$. Since $\mathbf{u} = \mathbf{f}(\mathbf{x}, \mathbf{u}^{\wedge}(\mathbf{x}))$, inequalities

(6.7) determine a certain domain $\|x\| \leqslant H_4$. To that end it is sufficient to single out in the domain $\|x\| \leqslant H_1$ all values of $x$ which satisfy the inequality $\|u\| = \|f(x, u^\wedge(x))\| \leqslant H_2$. Put $H^* = \min(H_3, H_4)$.

*Corollary.* Let condition 1 of Theorem 1 hold in the domain (6.7) and condition 2 in the domain $\|\xi^*\| \leqslant H_3$. Then $S = \{x_0: \|x(t; t_0, x_0)\| \leqslant H^*\}$ is the domain of attraction of the unperturbed motion $x = 0$ with respect to the variables $\xi, \eta$, i.e., $\|\xi^*(t; t_0, x_0)\| \to 0, t \to \infty$ for $x_0 \in S$.

*Remarks.* 1. It is assumed in the corollary that for all $x$, $u$ in the domain (6.7) the first $r$ rows of the matrix $F$ are linearly independent, and for all $\xi, \eta$ in the domain $\|\xi^*\| \leqslant H_3$ the same rows of the matrix $\Psi$ are linearly independent in that domain.

2. Thanks to the explicit form of the solutions $\xi$, $\eta$ of the closed system (2.1), (4.2), (5.1) no essential difficulties are involved in constructively determining the set $S$ by the method proposed here.

***7. The Choice of optimal control laws.*** The auxiliary linear system (4.3) was derived from the original, non-linear system by a non-linear transformation of variables. It follows that the behaviour of the variables $\rho_i, \rho_i^{\cdot}$, and consequently also that of the variables $x_{m+l+i}$ of the original system (2.1), (4.2), (5.1), is determined by system (4.3), (5.1), and the quality of the transient with respect to the variables $x_{m+l+i}$ in the non-linear system (2.1) may be controlled by varying the constants $a_i, b_i, c_i$ in the functionals (5.2) while solving the auxiliary optimal stabilization problem for system (4.3).

If condition 2 holds, then Eqs.(6.4) hold. As a result, by varying the values of the constants $a_i, b_i, c_i$ in (5.2) one can control the quality of the transient in the original system (2.1) not only with respect to the variables $x_{m+l+i}$, but also with respect to the variables (6.4). When that is done the resources necessary to create the controls $u_i$ in system (2.1) may be estimated using (4.2) and (5.1). Since $f(x, u^\wedge(x)) \equiv 0$ at $x^* = 0$, while the unperturbed motion $x = 0$ of system (2.1), (4.2) and (5.1) is stable with respect to $w$, we conclude that by adjusting the coefficients $a_i, b_i, c_i$ in (5.2) we can also regulate the control values $u_i$. In sum: the constants $a_i, b_i, c_i$ and the functional (6.3) may be chosen with a view to guaranteeing the desired quality of the transient in the non-linear starting system (2.1) with respect to the variables $\xi, \eta$, at a satisfactory cost in respect of the resources needed to create the controls $u_i$.

The ease with which each individual instance of the problem (corresponding to fixed values of $a_i, b_i, c_i$) can be solved justifies resorting to an iterative procedure in order to approximate a practically acceptable solution of problem (4.2), (5.1), and moreover the details of this procedure are based on functions obtained in closed form. When that is done the functional (6.3) is a generalized performance index of the control in the original, non-linear problem of stabilizing the unperturbed solution $x = 0$ of problem (2.1). This is in agreement with the role played by performance indices in the modern applied theory of automatic control.

***8. Stabilization of the equilibrium position of a rigid body using a Cardan-suspended gyroscope.*** Consider a free rigid body with principal central axes of inertia $Ox_1x_2x_3$, on which a Cardan-suspended gyroscope is mounted in such a way that the axis of its external gimbal is directed along $Ox_1$ and its fixed point coincides with the centre of mass of the body. The gyroscope is regulated by three motors, which generate torques about the axes of the external and internal gimbals and the axis of the rotor.

The equations of motion of this mechanical system are /11/:

$$
\begin{gathered}
A_1 q_1^{\cdot} = (A_2 - A_3) q_2 q_3 + u_1 \quad (1\ 2\ 3) \qquad (8.1)\\
\beta_{i1}^{\cdot} = q_3 \beta_{i2} - q_2 \beta_{i3} \quad (i = 1, 2, 3)\ (1\ 2\ 3)\\
A\alpha^{\cdot} = -(A_1 + A) q_1 - (A_2 + A) q_2 \sin\alpha \operatorname{tg}\beta + \\
(A_2 + A) q_3 \cos\alpha \operatorname{tg}\beta + K_1 + (K_2 \sin\alpha - K_3 \cos\alpha) \operatorname{tg}\beta\\
A\beta^{\cdot} = -(A_2 + A) q_2 \cos\alpha - (A_3 + A) q_3 \sin\alpha + \\
K_2 \cos\alpha + K_3 \sin\alpha\\
K_i = \sum h_k \beta_{ki}, \quad h_k^{\cdot} = 0 \quad (i, k = 1, 2, 3)\\
u_1 = -u_1^*, \quad u_2 = -u_1^* \sin\alpha \operatorname{tg}\beta - u_2^* \cos\alpha + u_3^* \sin\alpha \sec\beta\\
u_3 = u_1^* \cos\alpha \operatorname{tg}\beta - u_2^* \sin\alpha - u_3^* \cos\alpha \sec\beta
\end{gathered}
$$

Here $q_i\ (i = 1, 2, 3)$ are the projections of the instantaneous angular velocity vector on the axes $Ox_1x_2x_3$; $A_i\ (i = 1, 2, 3)$ are the moments of inertia of the body with respect to these axes, $A$ and $C$ are the equatorial and axial moments of inertia of the gyro, $\alpha$ and $\beta$ are angles defining the position of the gyro relative to the body, $\beta_{ik}$ are the direction cosines of the angles between the axes $Ox_1x_2x_3$ and the axes of the inertial system $OX_1X_2X_3$ and $K_i$ and $h_i\ (i = 1, 2, 3)$ are the projections of the angular momentum vector of the system on the axes

$Ox_1x_2x_3$ and $OX_1X_2X_3$, respectively. Eqs.(8.1) were derived on the assumption that $\cos\beta \neq 0$, i.e., the external and internal gimbals lie in different planes.

Eqs.(8.1) have the particular solution

$$q_i = 0,\quad \beta_{ik} = 1\ (i = k),\quad \beta_{ik} = 0\ (i \neq k)$$
$$K_i = h_i^\circ,\quad \alpha = \alpha_0,\quad \beta = \beta_0,\quad \gamma^\cdot = \gamma_0^\cdot\ (i,\ k = 1,\ 2,\ 3)$$
$$(\alpha_0,\ \beta_0,\ \gamma_0^\cdot,\ h_i^\circ - \text{const}) \tag{8.2}$$

which corresponds to the equilibrium position of the body in the inertial coordinate system ($\gamma_0^\cdot$ is the velocity of rotation of the gyro about itself). Taking $\beta_0 = 0$, i.e., assuming that at the start of the motion the planes of the internal and external gimbals are at right angles to one another, let us consider the problem of stabilizing the specified equilibrium position of the body through controls $u_i^*\ (i = 1, 2, 3)$ generated by the motors. To that end, introducing new variables $q_i' = q_i,\ \beta_{ik}' = \beta_{ik} - 1\ (i = k),\ \beta_{ik}' = \beta_{ik}\ (i \neq k)\ (i, k = 1, 2, 3), \alpha' = \alpha - \alpha_0$, $\beta' = \beta$ and returning to the original notation, we obtain the following equations of the perturbed motion:

$$A_1q_1^\cdot = (A_2 - A_3)\, q_2q_3 + u_1\ (1\ 2\ 3) \tag{8.3}$$
$$\beta_{ii}^\cdot = G_{ii},\quad \beta_{12}^\cdot = -q_3 + G_{12},\ \beta_{13}^\cdot = q_2 + G_{13}\ (L = 1,\ 2,\ 3)\quad (1\ 2\ 3)$$
$$A\alpha^\cdot = -(A_1 + A)\, q_1 - (A_2 + A)\, [q_2 \sin(\alpha + \alpha_0) - q_3 \cos(\alpha +$$
$$\alpha_0)]\operatorname{tg}\beta + h_1 + \sum (h_k^0 + h_k)\, \beta_{k1} + [G_1 \sin(\alpha + \alpha_0) - G_2 \cos(\alpha +$$
$$\alpha_0)]\operatorname{tg}\beta$$
$$A\beta^\cdot = -(A_2 + A)\, q_2 \cos(\alpha + \alpha_0) - (A_3 + A)\, q_3 \sin(\alpha + \alpha_0) +$$
$$G_1 \cos(\alpha + \alpha_0) + G_2 \sin(\alpha + \alpha_0)$$
$$h_i^\cdot = 0,\quad G_{i1} = q_3\beta_{i2} - q_2\beta_{i3}\quad (i = 1,\ 2,\ 3)\quad (1\ 2\ 3)$$
$$G_j = h_{1+j}^0 + h_{1+j} + \sum (h_k^0 + h_k)\, \beta_{k,\,j+1}\quad (j = 1,\ 2)$$
$$u_1 = -u_1^*,\quad u_2 = -u_1^* \sin(\alpha + \alpha_0)\operatorname{tg}\beta - u_2^* \cos(\alpha + \alpha_0) +$$
$$u_3^* \sin(\alpha + \alpha_0)\sec\beta,\quad u_3 = u_1^* \cos(\alpha + \alpha_0)\operatorname{tg}\beta -$$
$$u_2^* \sin(\alpha + \alpha_0) - u_3^* \cos(\alpha + \alpha_0)\sec\beta,\ \beta^\cdot_{ij} \triangleq \bar{G}_{ij}$$

To these equations we add the six relations

$$\beta_{ik} + \beta_{ki} + \sum \beta_{il}\beta_{kl} = 0\quad (i,\ k = 1,\ 2,\ 3;\ i \leqslant k) \tag{8.4}$$

implied by the relations between the direction cosines. Denote the expressions on the left of (8.4) by $Y_j\ (j = 1, \ldots, 6)$. Defining also $X_1^{(2)} = \bar{G}_{21},\ X_2^{(2)} = \bar{G}_{23},\ X_3^{(2)} = \bar{G}_{31}$ and introducing the notation $w_1 = \alpha,\ w_2 = \beta,\ \xi_i = q_i\ (i = 1,\ 2,\ 3),\ \eta_1 = \beta_{21},\ \eta_2 = \beta_{23},\ \eta_3 = \beta_{31}\ (m = 2,\ l = r = 3,\ k = 6,$ $n = 14)$, one can verify that system (8.3), (8.4) satisfies conditions 1 and 2 of Theorem 1.

The control laws (4.2) in this case are

$$u_s = \frac{A_s(Q_sA_2 + \beta_{2s}u_2)}{A_2(1 + \beta_{22})},\quad s = 1{,}3;\quad u_2 = \frac{A_2Q_2(1 + \beta_{22})}{(1 + \beta_{22})(1 + \beta_{33}) - \beta_{23}\beta_{32}} \tag{8.5}$$
$$Q_1 = Q_1^* - u_2^\wedge,\quad Q_3 = Q_3^* + u_1^\wedge$$
$$Q_2 = -(1 + \beta_{33})\, R_2 + \beta_{32}R_3 + q_3\bar{G}_{32} - q_2G_{33} - u_3^\wedge +$$
$$\beta_{32}Q_3/(1 + \beta_{22}),\quad A_1R_1 = (A_2 - A_3)\, q_2q_3\qquad (1\ 2\ 3)$$
$$Q_s^* = -(1 + \beta_{22})\, R_s + \beta_{2s}R_2 + q_2\bar{G}_{2s} - q_sG_{22},\ s = 1,\ 3$$

These controls laws are related to the controls $u_i^*\ (i = 1, 2, 3)$ generated by the motors as follows:

$$u_1^* = -u_1,\quad u_2^* = -u_2 \cos(\alpha +$$
$$\alpha_0) - u_3 \sin(\alpha + \alpha_0),\quad u_3^* = -u_1 \sin\beta +$$
$$[u_2 \sin(\alpha + \alpha_0) - u_3 \cos(\alpha + \alpha_0)]\cos\beta$$

The axuiliary linear system (4.3) consists of the equations $\beta_{21}^{\cdot\cdot} = u_1^\wedge,\ \beta_{23}^{\cdot\cdot} = u_2^\wedge,\ \beta_{31}^{\cdot\cdot} = u_3^\wedge$ and the "truncated" system (6.2) of the equations

$$A\alpha^\cdot = h_1 + [(h_2 + h_2^0)\sin(\alpha + \alpha_0) - (h_3 + h_3^0)\cos(\alpha + \alpha_0)]\operatorname{tg}\beta \tag{8.6}$$
$$A\beta^\cdot = (h_2 + h_2^0)\cos(\alpha + \alpha_0) + (h_3 + h_3^0)\sin(\alpha + \alpha_0)$$
$$h_i^\cdot = 0\quad (i = 1,\ 2,\ 3)$$

System (8.6) describes the motion of an uncontrolled gyro on an immobile base. In that case /11, 12/, if the initial perturbations of the angular momentum are small, the system will be stable will be stable for any finite (not necessarily large) angular velocity of rotation of the gyro in unperturbed motion. But if the initial perturbations of the angular momentum take arbitrary finite values, the stability condition will be observed only for sufficiently large values of $\gamma_0^{\cdot}$.

Thus, by the theorem proved above, the unperturbed motion of system (8.3)-(8.5) is stable in Lyapunov's sense and asymptotically stable with respect to the variables $q_i$, $\beta_{ik}$ $(i, k = 1, 2, 3)$. Incidentically, this stability of the unperturbed motion of the system with respect to $\alpha$, $\beta$ justifies our starting assumption $\cos\beta \neq 0$, $\beta_0 = 0$.

The control laws are small in mangitude if the initial perturbations of the angular momentum of the system are small. Optimal control laws can be chosen using the technique outlined in Sect.7. Eqs.(6.4) for our example are

$$\beta_{22} = -1 + \sqrt{1 - \beta_{21}^2 - \beta_{23}^2}, \quad \beta_{32} = [-\beta_{21}\beta_{31} - \beta_{23}(1 + \beta_{33})]/(1 + \beta_{22})$$
$$\beta_{33} = [-1 + \beta_{21}^2 - \beta_{21}\beta_{23}\beta_{31} + \sqrt{1 + M_1}]/(1 - \beta_{21}^2)$$
$$\beta_{12} = M_2/M^*, \quad \beta_{13} = M_3/M^*, \quad \beta_{11} = -1 + \sqrt{1 - \beta_{12}^2 - \beta_{13}^2}$$
$$M_1 = 2(1 + \beta_{22}) + (1 + \beta_{22})^2 - \beta_{31}^2 - 2(1 + \beta_{22})\beta_{31}^2 -$$
$$(1 + \beta_{22})\beta_{31}^2 - \beta_{21}^2 - 2(1 + \beta_{22})\beta_{21}^2 - (1 + \beta_{22})^2\beta_{21} -$$
$$\beta_{21}^2\beta_{31}^2 - 2(1 + \beta_{22})\beta_{21}^2\beta_{31}^2 - (1 + \beta_{22})\beta_{21}^2\beta_{31}^2$$
$$M_2 = -(1 + \beta_{33})\beta_{21} + \beta_{23}\beta_{31}, \quad M_3 = -(1 + \beta_{22})\beta_{31} - \beta_{21}\beta_{32}$$
$$M^* = \sqrt{M_2^2 + M_3^2 + M_4^2}, \quad M_4 = (1 + \beta_{22})(1 + \beta_{33}) - \beta_{23}\beta_{32}$$
$$q_1 = (-\beta_{23}^{\cdot} + q_2\beta_{21})/(1 + \beta_{22})$$
$$q_2 = [-\beta_{21}^{\cdot}\beta_{32} - (1 + \beta_{22})\beta_{31}^{\cdot}]/M_4$$
$$q_3 = [-\beta_{23}\beta_{31}^{\cdot} + \beta_{21}^{\cdot}(1 + \beta_{33})]/M_4$$

In the interests of clarity, we point out that the expressions for $q_i$ $(i = 1, 2, 3)$ were obtained by solving the equations $\beta_{21}^{\cdot} = \bar{G}_{21}$, $\beta_{23}^{\cdot} = \bar{G}_{23}$, $\beta_{31}^{\cdot} = \bar{G}_{31}$ for $q_i$, and those for $\beta_{ik}$ by solving system (8.4). As a result one obtains the explicit form of the solutions of the closed system (8.3)-(8.5), so that, by giving suitable values to $d_i$, $d_{i+1}$ in $u_i^{\wedge}$ $(i = 1, 2, 3)$, one can select control laws from the set (8.5) which are optimum in the practical sense - and this may be done in real time.

For a more detailed analysis of the objective functional (6.3), let us confine attention to the case $\alpha_0 = \beta_0 = 0$, that is, the planes of the internal and external gimbals in equilibrium are perpendicular to one another, and the axes of the external and internal gimbals and the rotor point along the appropriate principal central axes of inertia of the body. We transform from the interconnected variables $\beta_{ik}$ to the three independent Krylov angles /13/, which uniquely define the position of the body: $\varphi$ (yaw), $\psi$ (trim), $\theta$ (tilt), letting the $X_2$ and $x_3$ axes be the fundamental axes. Then

$$\beta_{12} = -\varphi + \ldots, \quad \beta_{13} = \psi + \ldots, \quad \beta_{21} = \varphi + \ldots, \quad \beta_{23} = -\theta + \ldots$$
$$\beta_{31} = -\psi + \ldots, \quad \beta_{32} = \theta + \ldots, \quad \beta_{21}^{\cdot} = q_3 + \ldots, \quad \beta_{23}^{\cdot} = -q_2 + \ldots$$
$$\beta_{31}^{\cdot} = -q_1 + \ldots, \quad u_1^{\wedge} = -u_3^*/A_3 + \ldots, \quad u_2^{\wedge} = u_1^*/A_1 + \ldots$$
$$u_3^{\wedge} = u_2^*/A_2 + \ldots$$

where the dots stand for terms of more than the first order in $q_i, \varphi, \psi, \theta, \alpha, \beta, u_i^*$ $(i = 1, 2, 3)$. In view of these relations we conclude that the integrand $P$ in the objective functional has the form

$$P = a_1\varphi^2 + a_2\theta^2 + a_3\psi^2 + b_1q_3^2 + b_2q_2^2 + b_3q_1^2 +$$
$$c_1u_3^{*2}/A_3^2 + c_2u_1^{*2}/A_1^2 + c_3u_2^{*2}/A_2^2 + P^*(\mathbf{q}, \varphi, \psi, \theta, \alpha, \beta, \mathbf{u}^*)$$

where the function $P^*$ contains terms of order three and above in $q_i$, $\varphi$, $\psi$, $\theta$, $\alpha$, $\beta$, $u_i^*$ $(i = 1, 2, 3)$.

Putting $b_1 = A_3$, $b_2 = A_2$, $b_3 = A_1$, we deduce that $P$ is the sum of four expressions: 1) the kinetic energy of the body $A_1q_1^2 + A_2q_2^2 + A_3q_3^2$; 2) a quadratic form $a_1\varphi^2 + a_2\theta^2 + a_3\psi^2$ characterizing the departure of the body from its equilibrium position $\varphi = \psi = \theta = 0$; 3) a quadratic form $c_2u_1^{*2}/A_1^2 + c_3u_2^{*2}/A_2^2 + c_1u_3^{*2}/A_3^2$ characterizing the resources expended in generating the controls; 4) a function $P^* = P^*(\mathbf{q}, \varphi, \psi, \theta, \alpha, \beta, \mathbf{u}^*)$, defined during the solution.

Technically speaking, the stabilization laws thus obtained are implemented as follows. When the body departs from equilibrium, special devices measure $q_i$, $\beta_{ik}$ $\alpha$, $\beta$ and send appropriate signals to the controlling motors, which respond by generating torques $u_i^*$ $(i=1, 2, 3)$ and applying them to the axes of the Cardan suspension. As a result of the torques $u_i^*$ $(i = 1, 2, 3)$, the body is restored to its original equilibrium position.

***9. Generalization of the main result.*** We shall show that the possibilities of constructing the auxiliary linear system (4.3) may be extended if one considers, instead of (4.1), equalities

$$X_j^{(1)}(\mathbf{x},\ \mathbf{u}) = u_j^\wedge\ (j = 1,\ \ldots,\ l_1 < r) \tag{9.1}$$
$$\Phi_i(\mathbf{x},\ \mathbf{u}) = u_{l_1+i}^\wedge\ (i = 1,\ \ldots,\ n - m - l) \tag{9.2}$$

and instead of the conditions $n - m - l \geqslant r,\ k \geqslant n - m - 2r$ one requires that $n - m - l \geqslant r - l_1,\ k \geqslant n - m - 2r + l_1$.

Let $F^*(\mathbf{x},\ \mathbf{u})$ be the Jacobian of the functions $X_j^{(1)}\ (j = 1, \ldots, l_1)$, $\Phi_j\ (j = 1, \ldots, n - m - l)$ with respect to the variables $u_i$. Let rank $F^*(0, 0) = r$ and, without loss of generality, let us assume that the linearly independent rows in $F^*(0, 0)$ are the first $r$ ones. Let $\Psi^*(\mathbf{x})$ be the Jacobian of the functions $X_i^{(2)}\ (i = 1, \ldots, r - l_1)$, $Y_i\ (i = 1, \ldots, k)$ with respect to the variables $\xi_{l_1+1}, \ldots, \xi_l, \eta_{r-l_1+1}, \ldots, \eta_{n-m-l}$. Since rank $F^*(0, 0) = r$, it follows from the theory of implicit functions that in the neighbourhood of the point $\mathbf{x} = 0,\ \mathbf{u} = 0$, for all $u_i^\wedge$, there exits a solution

$$\mathbf{u} = \mathbf{f}^*(\mathbf{x}, \mathbf{u}^\wedge),\quad \mathbf{u}^\wedge = (u_1^\wedge, \ldots, u_r^\wedge) \tag{9.3}$$

of system (9.1), (9.2), where the vector-valued function $\mathbf{f}^*$ is continuously differentiable in the neighbourhood of $\mathbf{x} = 0$ for all $u_i^\wedge$ and $\mathbf{f}^*(0, 0) = 0$.

It can be shown by a direct calculation that there is a neighbourhood of the point $\mathbf{x} = 0$, $\mathbf{u} = 0$ in which the closed system (2.1), (9.3) splits off a system of linear equations

$$\dot\rho_i = u_i^\wedge\quad (i = 1,\ \ldots,\ l_1 < r) \tag{9.4}$$
$$\ddot\rho_{l_1+j} = u_{l_1+j}^\wedge\quad (j = 1,\ \ldots,\ r - l_1) \tag{9.5}$$

It can be shown by dynamic programming methods that if

$$u_i^\wedge = -\sqrt{a_i^*/c_i^*}\rho_i\quad (i = 1,\ \ldots,\ l_1) \tag{9.6}$$

then the equilibrium position $\rho_i = 0\ (i = 1, \ldots, l_1)$ of each of the subsystems (9.4) is asymptotically stable in Lyapunov's sense and a functional of the form

$$I_i^* = \int_{t_0}^{\infty} [a_i^*\rho_i^2 + c_i^* u_i^{\wedge 2}]\, dt$$

is minimized on the trajectories of each subsystem.

*Theorem 2.* Assume that the following conditions hold:

1°. rank $F^*(0,\ 0) = r$; 2°. rank $\Psi^*(0) = n - m - r$

Let the trivial solution $\mathbf{w} = 0$ of the "truncated" system (6.2) be stable in Lyapunov's sense. Then the unperturbed solution $\mathbf{x} = 0$ of the closed system (2.1), (9.3), (5.1), (9.6) (Eqs.(5.1) hold for $i = l_1 + 1, \ldots, r$) is stable in Lyapunov's sense and asmyptotically stable with respect to $\xi, \eta$. The functional

$$I^* = \int_{t_0}^{\infty} \Big\{ \sum_{i=1}^{l_1} [a_i^* x_{m+i}^2 + c_i^*(X_i^{(1)}(\mathbf{x}, \mathbf{u}))^2] + \sum_{j=1}^{r-l_1} [a_j x_{m+l+j}^2 + b_j(X_j^{(2)}(\xi, \eta))^2 + c_j\Phi_j^2(\mathbf{x}, \mathbf{u})]\Big\}\, dt \tag{9.7}$$

is minimized on the trajectories of this system.

Using the technique developed in Sect.7, the constants $a_j,\ b_j,\ c_j\ (j = 1, \ldots, r - l_1)$, $a_i^*$, $c_i^*\ (i = 1, \ldots, l_1)$ in (9.7) can be selected so as to achieve the desired quality of the transient in system (2.1),(9.3),(5.1),(9.6) with respect to the variables $\xi, \eta$, at a satisfactory cost in the resources needed to generate the controls $u_i$.

*Remarks.* 1. In the case $m = 0$ (when system (2.1) consists of the equations $\dot\xi = \mathbf{X}^{(1)}(\xi, \eta, \mathbf{u})$, $\dot\eta = \mathbf{X}^{(2)}(\xi, \eta)$), if conditions 1 and 2 are satisfied, the unperturbed solution of systems (2.1), (4.2), (5.1) and (2.1), (9.3), (5.1), (9.6) are asymptotically stable in Lyapunov's sense, by Theorems 1 and 2, respectively.

2. As in Sect.7, there is a constructive procedure to determine the domain of attraction of the unperturbed motion $\mathbf{x} = 0$ with respect to the variables $\xi, \eta$.

3. Theorems 1 and 2 extend the results of /7, 8/ to a broader range of non-linear systems.

*Example.* Consider Euler's dynamic equations

$$A_1\dot q_1 = (A_2 - A_3)\, q_2 q_3 + u_1\ (1\ 2\ 3) \tag{9.8}$$

which describe the rotational motion of a rigid body about its centre of inertia $O$. Here $q_i$

are the projections of the angular velocity vector of the body on its principal central axes of inertia $Ox_1x_2x_3$. $A_i$ are the principal central moments of inertia and $u_i$ are the controlling torques, $i = 1, 2, 3$.

Let **s** be a unit vector, with projections $s_i$ $(i = 1, 2, 3)$ on the axes $Ox_1x_2x_3$, representing a specific orientation in inertial space; then

$$s_1^{\cdot} = s_2q_3 - s_3q_2 \quad (1\ 2\ 3) \tag{9.9}$$

Eqs.(9.8) and (9.9) have a particular solution

$$q_i = u_i = 0 \quad (i = 1, 2, 3), \quad s_1 = s_3 = 0, \quad s_2 = 1 \tag{9.10}$$

corresponding to the equilibrium position of the body when there are no controlling torques, on the assumption that one of the principal central axes of inertia of the body (the $x_2$ axis) points along the vector **s**.

Consider the problem of stabilizing the equilibrium position (9.10) by controls $u_i$ $(i = 1, 2, 3)$. To that end, introducing new variables $q_i' = q_i$ $(i = 1, 2, 3)$, $s_j' = s_j$ $(j = 1, 3)$, $s_2' = s_2 - 1$ and returning to the original notation, we construct the system of equations of the perturbed motion:

$$A_1q_1^{\cdot} = (A_2 - A_3)\,q_2q_3 + u_1 \quad (1\ 2\ 3) \tag{9.11}$$

$$s_1^{\cdot} = (s_2 + 1)\,q_3 - s_3q_2 \triangleq \varphi_1, \quad s_2^{\cdot} = s_3q_1 - s_1q_3 \triangleq \varphi_2$$

$$s_3^{\cdot} = s_1q_1 - (s_2 + 1)\,q_1 \triangleq \varphi_3, \quad s_1^2 + (s_2 + 1)^2 + s_3^2 = 1$$

Setting $\xi_i = q_i$, $\eta_i = s_i$ $(i = 1, 2, 3)$, $m = 0$, $Y_1 = s_1^2 + (s_2 + 1)^2 + s_3^2 - 1$, we can verify that the structure of system (9.11) is just that of the second and third group of Eqs.(2.1). However, condition 1 of Theorem 1 fails to hold. At the same time, setting

$$\xi_i = q_i \ (i = 1, 2, 3), \quad \eta_1 = s_1, \ \eta_2 = s_3, \quad \eta_3 = s_2$$

$$A_2X_1^{(1)} = (A_3 - A_1)\,\xi_1\xi_3, \qquad X_1^{(2)} = (\eta_2 + 1)\,\xi_3 - \eta_3\xi_2, \qquad X_2^{(2)} = \eta_1\xi_2 - (\eta_2 + 1)\,\xi_2$$

we conclude that conditions 1 and 2 of Theorem 2 hold for system (9.11).

The control laws (9.3) in this case are

$$u_j = \frac{A_j\,(A_2f_j + s_ju_2)}{A_2\,(s_2 + 1)}, \quad j = 1, 3; \quad u_2 = A_2\,(u_1^{\wedge} - \psi_2) \tag{9.12}$$

$$f_1 = f_1^* - u_1^{\wedge}, \quad f_3 = f_3^* + u_2^{\wedge}$$

$$f_j^* = \psi_2s_j - \psi_j\,(s_2 + 1) + \varphi_jq_2 - \varphi_2q_j, \quad j = 1, 3$$

$$A_1\psi_1 = (A_2 - A_3)\,q_2q_3 \qquad (1\ 2\ 3)$$

and the auxiliary system (9.4), (9.5) consists of the equations $q_2^{\cdot} = u_1^{\wedge}$, $s_1^{\cdot\cdot} = u_2^{\wedge}$, $s_3^{\cdot\cdot} = u_3^{\wedge}$. Making the appropriate choice of $u_i^{\wedge}$ $(i = 1, 2, 3)$, we conclude that the unperturbed motion of system (9.11), (9.12) is asymptotically stable in Lyapunov's sense.

We note that these control laws may solve, for example, applied problems such of stabilizing the orientation of a spacecraft in interplanetary flight by brief orientation sessions /14/, a problem which in many cases reduces to the above problem of stabilizing the equilibrium position of a rigid body in inertia space. Various other problems of controlling the angular motion of rigid bodies (such as the reorientation and stabilization on an orbit) are analysed using the technique proposed here in /15, 16/.

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# MAXIMALLY FAST BRAKING OF AN OBJECT IN CONTROLLED MOTION, UNDER THE ACTION OF AERODYNAMIC DRAG AND GRAVITY*

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In order to study controlled motion of objects in the atmosphere, which travel under the influence of aerodynamic drag and gravity, a model problem is used to investigate the mechanism by which these forces affect the intensity of braking of the object in an exponential atmosphere. A time-optimal control is synthesized for objects whose aerodynamic drag may be characterized exclusively as a force proportional to the product of the velocity of motion times the density of the atmosphere at the current altitude of motion, on the assumption that the atmosphere is exponential. The control synthesis, represented in generalized coordinates $V, \rho$, is independent of the braking characteristic $T$ of the object and the parameter $k$ characterizing the variation of atmospheric density; it is determined solely by the magnitude of the generalized terminal stopping velocity $V_k$ and the values of $\rho_{\min/\max}$, which are determined by the position of the boundaries of the phase constraint (PC). It is shown by a numerical experiment how one can simplify the optimal synthesis by introducing a certain control "significance" level.

*1. Statement of the problem.* In connection with the control of motion in an exponential atmosphere (i.e. /1/, the density of the atmosphere varies exponentially with altitude $H$: $r = r_0 \exp(-kH)$), in a plane-parallel gravitational field, we shall consider the mechanism by which aerodynamic drag and the force of gravity affect the braking of an object. The model problem studied below may be given the following physical interpretation.

In an inertial coordinate system (see Fig.1, in which $XOZ$ is the plane of the local horizon and the $H$ axis points along the gravitational lines of force), an object $P$ (the pursuer) moving at velocity $V_p$ is approaching an object $Z$ (the pursued object) which is moving at a velocity $V_z$. Initially, the object $P$ receives a starting impulse which imparts to it an initial velocity $V_{p0}$. As it continues to move, its mass remains constant, but the direction of its velocity $V_p$ is modified by its control system, which ensures that the projections of $V_p$ and $V_z$ on a plane perpendicular to the line joining $P$ and $Z$ (called the range line, $CPZ$) remain equal to all times. The magnitude of $V_p$ decreases under the action of aerodynamic drag, whose magnitude may be considered proportional to the product $r_p V_p$

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